High Field Science with Lasers

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$E_s = \frac{m_e c^2}{e \lambda_c} = \frac{m_e^2 c^3}{e \hbar}$

Nonlinear QED: $E \cdot e \cdot \lambda_c = 2m_0 c^2$

Zettawatt Laser

Laser Intensity Limit: $I = \frac{\hbar v^3}{c^2} \cdot \frac{\Delta v_g}{\sigma} = \frac{P_{th}}{\lambda^2}$

Relativistic Optics: $v_{osc} \sim c$

Bound Electrons: $E = \frac{e^2}{a_0}$

Electron Characteristic Energy

Extreme Field Limits: Nonlinear QED Vacuum

Towards studying of nonlinear QED effects with high power lasers. Radiation dominated & QED regimes in the high intensity electromagnetic wave interaction with charged particles & vacuum.

\[ E_S = \frac{m_e c^3}{e \hbar} \quad I_S = c \frac{E_S^2}{4\pi} \approx 10^{29} \frac{W}{cm^2} \]

Schwinger (Sauter, Bohr,…) field
The critical electric field of quantum electrodynamics, called also the Schwinger field, is so strong that it produces electron-positron pairs from vacuum, converting the energy of light into matter. Since the dawn of quantum electrodynamics there has been a dream on how to reach it on Earth. With the rise of the laser technologies this field becomes feasible through construction of extreme power lasers or/and with the sophisticated usage of nonlinear processes in relativistic plasmas. This is one of the most attractive motivations for extreme power laser development, i.e. producing matter from vacuum by pure light in fundamental process of quantum electrodynamics in the non-perturbative regime.

The laser with intensity well below the Schwinger limit can create an avalanche of electron-positron pairs similar to a discharge before attaining the Schwinger field. It was also realized that the Schwinger limit can be reached using an appropriate configuration of laser beams.

The regimes of dominant radiation reaction, which completely changes the electromagnetic wave-matter interaction, will be revealed. This will result in a new powerful source of high brightness gamma-rays. A possibility of the demonstration of the electron-positron pair creation in vacuum in a multi-photon processes can be realized. This will allow modeling under terrestrial laboratory conditions the processes in various astrophysical objects.

An ultra-relativistic electron emits Cherenkov radiation in vacuum with an induced by strong electromagnetic wave refraction index larger than unity. During the interaction with this wave the electron also radiates photons via the Compton scattering. Synergic Cherenkov-Compton process can be observed by colliding laser accelerated electrons with a high intensity electromagnetic pulse.
Extreme Field Limits

non-perturbative QED
$10^{29}$
nonlinear - dissipative
vacuum

multiphoton QED processes
$10^{27}/\nu_e^2$

lepton - gamma plasma
$10^{24}/\nu_e^2$

radiation friction
$10^{23}/\nu_e^{2/3}$
$\chi, \gamma$ - rays

relativistic nonlinear optics
$10^{18}$
plasma

$E_0 < e^2/a_0 = m_e^2 e^5/h^2$

https://www.eli-beams.eu/projekty/hifi/
FOR READING

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References


Laser-matter interaction transition from the relativistic regime to dominant radiation friction and nonlinear Thomson scattering through the limit when quantum electrodynamics comes into play and to electron-positron avalanche/cascade development, towards nonlinear vacuum with electron-positron creation and nonlinear vacuum polarization while the EM field intensity increases.
1. Normalized EM wave amplitude

\[ a = \frac{e \sqrt{\left( A_\mu \right)^2}}{m_e \omega_c} = \frac{eE\lambda}{2\pi m_e c^2} \quad a = 1 \quad \Leftrightarrow \quad I_{rel} = \frac{m^2 c^3 \omega^2 a^2}{4\pi e^2} = 1.37 \times 10^{18} \left( \frac{1\mu m}{\lambda} \right)^2 \frac{W}{cm^2} \]

2. Radiation friction becomes important for

\[ a > a_{rad} = \varepsilon^{-1/3}_{rad} \quad \text{with} \quad \varepsilon_{rad} = \frac{4\pi r_e}{3\lambda} \quad a = a_{rad} \quad \Leftrightarrow \quad I_R = \frac{3m^8 c^5 \omega^{4/3}}{8\pi e^{10/3}} = 2.65 \times 10^{23} \left( \frac{1\mu m}{\lambda} \right)^{4/3} \frac{W}{cm^2} \]

where \( r_e = e^2 / m_e c^2 \) - classical electron radius

3. QED parameters

\[ \chi_e = \frac{e\hbar \sqrt{\left( F^{\mu\nu} p_\mu \right)^2}}{m_e^3 c^4} \approx \frac{E}{E_S} \frac{p_e}{m_e c} \quad \text{and} \quad \chi_\gamma = \frac{e\hbar^2 \sqrt{\left( F^{\mu\nu} k_\mu \right)^2}}{m_e^3 c^4} \approx a \frac{\hbar^2 \omega_o \omega_\gamma}{m_e^2 c^4} \quad \left[ N\omega_0 + \omega_\gamma \rightarrow e^+ e^- \right] \]

4. QED limit:

\[ \chi_e \approx 1 \quad \Rightarrow \quad I_Q = \frac{m^3 c^5 \omega}{8\pi e^2 \hbar} = 5.75 \times 10^{23} \left( \frac{1\mu m}{\lambda} \right) \frac{W}{cm^2} \]

\[ E_S = \frac{m^2 c^3}{e\hbar} \quad \Leftrightarrow \quad a_S = \frac{m_e c^2}{\hbar \omega} \quad a = a_S \quad \Leftrightarrow \quad I_S = \frac{m^4 c^7}{4\pi e^2 \hbar^2} = 2.36 \times 10^{29} \frac{W}{cm^2} \]
Electron Motion in High Intensity EM Wave

When the recoil of the emitted photon is significant, the emission probability is characterized by $\chi_e$ parameter (Lorentz and gauge invariant)

$$\chi_e = \left(\frac{\gamma_e}{E_S}\right)\left[(\mathbf{E} + \mathbf{\beta} \times \mathbf{B})^2 - (\mathbf{\beta} \cdot \mathbf{E})^2\right]^{1/2}$$

At $\chi_e > \chi_e^* \approx 1$ the QED effects come into play.

Dimensionless amplitude

$$a = \frac{eE}{m_e \omega c}$$

At $a=a_{rad}$ emitted energy becomes equal to the energy received from EM wave.

$$a_{rad} = \left(\frac{3\lambda}{4\pi r_e}\right)^{1/3} \quad r_e = \frac{e^2}{m_e c^2}$$
Curves $I_R(w)$, $I_Q(w)$ and $I_{R-Q}(w)$, $I_{Q-R}(w)$ subdivide $(I, w)$ plane to 4 domains:

I) Relativistic electron - EM field interaction with neither radiation friction nor QED effects

II) Electron - EM wave interaction is dominated by radiation friction

III) QED effects important with insignificant radiation friction effects

IV) Both QED and radiation friction determine radiating charged particle dynamics in the EM field

$\omega_i = \frac{e^4 m_e}{18 \hbar^4}$

$\lambda_i \approx 820 \text{ nm}$

$87 \frac{c^5 \hbar^8 \omega^4}{e^{14}} = 8.2 \times 10^{21} \left( \frac{\omega}{\omega_i} \right)^4 \frac{W}{\text{cm}^2}$
Radiation Friction Force:

Lorentz-Abramham-Dirac, Pomeranchuk, Landau-Lifshitz, ...
Minkovski equations: \[ \frac{du^\mu}{ds} = \frac{e}{m_e c} F_{\nu}^\mu u^\nu \] where \[ u^\mu = \frac{dx^\mu}{ds}, \quad ds = \frac{dt}{\gamma} \]

EM field tensor: \[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

Maxwell equations: \[ \varepsilon^{\mu\nu\rho\sigma} \partial_\rho F_{\nu\sigma} = 0 \quad \text{and} \quad \partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \]

In 3D notations: \[ \dot{p} = e \left( E + \frac{1}{c} v \times B \right), \quad \dot{x} = c \frac{p}{m_e \gamma}, \quad \gamma = \left( 1 + \frac{p_\mu p^\nu}{m_e^2 c^2} \right)^{1/2} \]

\[ \nabla \times B = \frac{4\pi}{c} j + \frac{1}{c} \partial_t E \]

\[ \nabla \cdot B = 0 \]

\[ \nabla \times E = -\frac{1}{c} \partial_t B \]

\[ \nabla \cdot E = 4\pi \rho \]
Intensity of radiation emitted by electron is given by

\[ I = \frac{2e^2}{3m_e^2 c^3} \left( dp_i \, dp_i \right) \]

In circularly polarized EM wave (in plasma), whose amplitude is equal to electron energy losses are

\[ \hat{E}^{(-)} = \frac{2e^4 E_0^2}{3m_e^2 c^3} \left[ 1 + \left( \frac{eE_0}{m_e \omega_0 c} \right)^2 \right] \]

For linearly polarized wave we have

\[ \hat{E}^{(-)} = \frac{e^4 E_0^2}{3m_e^2 c^3} \left[ 1 + \frac{3}{8} \left( \frac{eE_0}{m_e \omega_0 c} \right)^2 \right] \]

Pattern of field emitted by electron. T.Shintake, 2003

L.D.Landau & E.M.Lifshitz
“The Classical Theory of Fields”
Equations of electron motion are:

\[ m_e c^2 \frac{du^\mu}{ds} = \frac{e}{c} F^{\mu\nu} u_\nu + g^\mu \]

Radiation friction force is given by

\[ g^{\mu} = \frac{2e^2}{3c} \left( \frac{d^2 u^\mu}{ds^2} - u^\mu u^\nu \frac{d^2 u_\nu}{ds^2} \right) \]

Here \( \mu = 0, 1, 2, 3 \), \( s \) is proper time: \( ds = c \, dt / \gamma \)

4-velocity is

\[ u^i = \frac{dx^i}{ds} = \left( \gamma, \frac{p}{m_e c} \right) \]

and \( F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \) is 4-tensor of EM field
Covariant and 3D forms of L-L expression

Radiation reaction force in the Landau-Lifshitz approximation (L-L II, §76):

\[
m_e c^2 \frac{du^{\mu}}{ds} = \frac{e}{c} F^{\mu\nu} u_{\nu} + \frac{2e^3}{3m_e c^3} \left\{ \partial_\lambda F^{\mu\nu} u_{\nu} u_\lambda - \frac{e}{m_e c^2} \left[ F^{\mu\lambda} F_{\nu\lambda} u^{\nu} - (F_{\nu\lambda} u^{\nu})(F^{\nu\kappa} u_\kappa) u^\mu \right] \right\}
\]

\[
m_e c^2 \frac{dp}{dt} = e \left( E + \frac{v}{c} \times B \right) + \frac{2e^3}{3m_e c^3} \gamma^2 \left\{ (\partial_t + v \cdot \nabla) E + \frac{1}{c} v \times ((\partial_t + v \cdot \nabla) B) \right\} + \frac{2e^4}{3m_e^2 c^4} \left\{ E \times B + \frac{1}{c} B \times (B \times v) + \frac{1}{c} E (v \cdot E) \right\} - \frac{2e^4}{3m_e^2 c^5} \gamma^2 v \left\{ \left( E + \frac{v}{c} \times B \right)^2 - \frac{1}{c^2} (v \cdot E)^2 \right\}
\]

\[
\propto \varepsilon_{\text{rad}} \gamma^2 a_0
\]

Dimensionless parameter:

\[
\varepsilon_{\text{rad}} = \frac{2e^2 \omega}{3m_e c^3} = \frac{4\pi r_e}{3\lambda} \approx 10^{-8} \left( \frac{1\mu m}{\lambda} \right)
\]

Weak field approximation:

\[
E < E_{cl} = \frac{e}{r_e^2} = \frac{m_e c^4}{e^3} = \frac{1}{\alpha} E_S, \quad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}
\]
We consider the electron colliding with the EM wave given by
\[ a_0(t - x/c) \approx a_0(2t) \]

Retaining the main order terms in the L-L radiation friction force, we obtain equation for the \( x \)-component of the electron momentum
\[
\frac{d\gamma}{dt} = -\varepsilon_{rad} \omega a_0^2(2t)\gamma^2, \quad \varepsilon_{rad} = \frac{2r_e}{3\lambda}
\]

Its solution is
\[
\gamma(t) = \frac{\gamma_0}{1 + \varepsilon_{rad} \omega \gamma_0 \int_0^t a_0^2(2t')dt'}, \quad \gamma \rightarrow \frac{1}{\varepsilon_{rad} \omega \tau_{las} a_0^2}
\]

For \( \varepsilon_{rad} = 10^{-8} \quad \omega \tau_{las} = 100 \quad a_0 = 300 \), the electron gamma-factor becomes equal to \( \gamma = 10 \)

Radiation friction effects on charged particle motion

Equations of electron motion

\[ m_\text{e} c^2 \frac{du^\mu}{ds} = \frac{e}{c} F'^{\mu\nu} u_\nu + g^\mu \]

with radiation friction force \( g^\mu \)

EM wave is modelled by rotating \( E \) field

\[ \dot{q} = -a - \frac{\epsilon_{\text{rad}} G_e (\chi_e)}{\gamma_e} \left\{ \gamma^2 \dot{a} - a (q \cdot a) + q \left[ (\gamma a)^2 - (q \cdot a)^2 \right] \right\} \]

\[ n = \frac{n}{n_{\text{cr}}}, \quad \tau = \Omega t, \quad q = \frac{p}{m_\text{e} c}, \quad a = \frac{eE}{m_\text{e} \Omega c}, \quad \gamma_e = \left( 1 + q_1^2 + q_2^2 + q_3^2 \right)^{1/2} \]

QED effects incorporated with the form-factor, \( G_e (\chi_e) \), equal to the ratio of the full radiation intensity to the intensity emitted by a classical electron

\[ G_e (\chi_e) = -\frac{3}{4} \int_0^\infty \left[ \frac{4 + \chi_e x^{3/2} + 4 \chi_e^2 x^3}{\left( 1 + \chi_e x^{3/2} \right)^4} \right] \Phi' (x) x dx \]

where \( \Phi(x) \) is the Airy function

In order to describe the electron motion we write the electron momentum as

\[
\begin{pmatrix}
q_1 \\
q_{\parallel} \\
q_{\perp}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos(\tau) & \sin(\tau) \\
0 & -\sin(\tau) & \cos(\tau)
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix}
\]

Here \(q_{\parallel}\) and \(q_{\perp}\) are the components of the electron momentum parallel and perpendicular to the electric field.

We assume here that the wave is given.

Neglecting the change of the \(q_1\) component (near-critical plasma density), we obtain

\[
\dot{q}_{\perp} - q_{\parallel} = -\varepsilon_{\text{rad}} G_e(\chi_e) \left[ \gamma_e a + a^2 \frac{q_{\perp}}{\gamma_e} \left( 1 + q_{\perp}^2 \right) \right]
\]

\[
\dot{q}_{\parallel} + q_{\perp} = a - \varepsilon_{\text{rad}} G_e(\chi_e) a^2 q_{\parallel} \frac{q_{\perp}^2}{\gamma_e}
\]

with the energy balance equation

\[
\dot{\gamma}_e = a u_{\parallel} - \varepsilon_{\text{rad}} G_e(\chi_e) \left( a q_{\perp} + a^2 q_{\perp}^2 \right)
\]

The QED parameter \(\chi_e\) is given by

\[
\chi_e = \frac{a}{a_s} \sqrt{1 + q_{\perp}^2}
\]

Dependence of $q_2$ and $q_3$ (a,b), $q_{\perp}$ and $q_{\parallel}$ (c,d) and $G_e$ and $\chi_e$ (e,f) on time for $\varepsilon_{\text{rad}} = 10^{-8}$ and $a_s = 4.1 \times 10^3$

(a,c) $a = 0.25 \varepsilon_{\text{rad}}^{-1/3}$, (d,e) $a = 0.75 \varepsilon_{\text{rad}}^{-1/3}$. 
**Stationary solutions**

The energy flux reemitted by the electron is equal to $e(v \cdot E)$, which is $\varepsilon_{rad} G_e(\chi_e) m_e c^2 \omega e \left( a q_\perp + a^2 q_\perp^2 \right)$.

The integral scattering cross section by definition equals the ratio of the reemitted energy flux to the Poynting vector magnitude:

$$\sigma = \sigma_T G_e(\chi_e) \left( \frac{q_\perp}{a} + q_\perp^2 \right)$$

Here $\sigma_T$ is the Thomson scattering cross section $\sigma_T = \frac{8\pi r_e^2}{3} = 6.65 \times 10^{-25}$ cm$^2$

More convenient formula is

$$\sigma = \sigma_T \left( \frac{G_e \gamma_e^2}{1 + \varepsilon_{rad}^2 G_e^2 \gamma_e^6} \right)$$

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Ya. B. Zel'dovich, Sov. Phys. Usp. 18, 97 (1975)
Dependences of \( \lg \left( \sigma / \sigma_T \right) \) and of \( \lg \gamma_e \) on \( \lg a \)
1) \( \omega = \omega_l / 12.5 \), 2) \( \omega = \omega_l \), 3) \( \omega = 12.5 \omega_l \).

Nonlinear Thomson - Compton Scattering Cross Section and Electron Gamma Factor vs EM Field Amplitude.
High Power Gamma Flare Generation
High Power Gamma-Ray Source

Concept of high power gamma-flash generation:

Energy of photons emitted via NTS scales as $\mathcal{E}_\gamma = \hbar \omega_\gamma \approx \hbar \omega e^3$.

Electron energy is of the order of $m_e c^2 a$.

Photon energy is in the $\gamma$-ray range if $a > 10^2$ with intensity above $10^{22} \text{ W/cm}^2$.

Applications

- Photo-nuclear reactions
- Electron-positron pair creation
- Medicine
- Material sciences
- ............
- Radiation safety

Photo-fission cross-section and pair production cross-section in uranium


Energy of emitted photon is

\[ \hbar \bar{\omega}_\gamma = 0.3 \hbar \bar{\omega}_0 a_0^3 \]

Number of emitted photons/wave period

\[ N_\gamma \approx \frac{3\pi}{4} \alpha a_0 \]
Density of aluminum target (g/cm$^3$) after 1 ns of irradiation by prepulse with a 2 μm spot FWHM size, and intensity $2 \times 10^{11}$ W/cm$^2$. Below is density along x-axis (solid curve) and initial target profile (dashed curve).

Density of iron target (g/cm$^3$) after 3 ns of irradiation by prepulse with a 2 μm spot FWHM size, and intensity $2 \times 10^{11}$ W/cm$^2$. Below is density along x-axis (solid curve) and initial target profile (dashed curve).

Optimal conditions for gamma flare generation (Hydrogen)

$10PW \quad \tau_{las} = 150\ fs \quad \lambda = 1\ \mu m \quad \varnothing = 2.5\ \mu m$

$L_{pl} = 80\ \mu m \quad (ELI-BL\ L4)$

- Electron density and laser intensity
- Electron x-px phase
- Ion x-px phase.

$t = 180\ fs, \ < 1\ %\ in\ gamma\ flare$

$t = 400\ fs, \ 36\ %\ in\ gamma\ flare$

Solid hydrogen (Z=1)

- Pulse self-focusing, hole boring, electron heating, and ion acceleration
Optimal conditions for gamma flare generation (Gold target)

10PW $\tau_{\text{las}} = 150\text{fs}$ $\lambda = 1\mu m$ $\varnothing = 2.5\mu m$

$L_{pl} = 80\mu m$ (ELI-BL L4)

- Electron density and laser intensity
- Electron x-px phase
- Ion x-px phase.

$t = 180\text{fs}$, < 1% in gamma flare

$t = 400\text{fs}$, 36% in gamma flare

Gold target (Z=67)

- Electron density and laser intensity
- Electron x-px phase
- Ion x-px phase.

Pulse self-focusing, hole boring, electron heating, and ion acceleration

E < 10MeV
E > 10MeV
E > 100MeV

Energy (MeV)

Time (fs)
Optimal conditions for gamma flare generation (Iron target)

\[
10 PW \quad \tau_{\text{las}} = 150 \, \text{fs} \quad \lambda = 1\, \mu \text{m} \quad \varnothing = 2.5\, \mu \text{m}
\]

\[
L_{\text{pl}} = 80\, \mu \text{m} \quad \text{(ELI-BL L4)}
\]

- Electron density and laser intensity
- Electron x-px phase
- Ion x-px phase.

\( t = 180 \, \text{fs}, \quad \text{< 1\% in gamma flare} \)
\( t = 400 \, \text{fs}, \quad \text{36\% in gamma flare} \)

Iron target (Z=24)

- Pulse self-focusing, hole boring, electron heating, and ion acceleration

\[
\begin{align*}
E < 10\, \text{MeV} \\
E > 10\, \text{MeV} \\
E > 100\, \text{MeV}
\end{align*}
\]

- Electron
- Photon

\[
\begin{align*}
\text{Energy (a.u.)} & \\
\text{Time (fs)}
\end{align*}
\]
Optimal conditions for gamma flare generation (Tilted Iron target)

10PW $\tau_{\text{las}} = 150\,\text{fs}$ $\lambda = 1\,\mu\text{m}$ $\varnothing = 2.5\,\mu\text{m}$

$L_{pl} = 80\,\mu\text{m}$ (ELI-BL L4)

- Electron density and laser intensity
- Electron $x$-px phase
- Ion $x$-px phase.

$t = 180\,\text{fs}$, < 1% in gamma flare
$t = 400\,\text{fs}$, 36% in gamma flare

Tilted iron target (Z=24)

- Electron density and laser intensity
- Electron $x$-px phase
- Ion $x$-px phase.

Pulse self-focusing, hole boring, electron heating, and ion acceleration

E < 10MeV
E > 10MeV
E > 100MeV

0.0 0.2 0.4 0.6 0.8 1.0

0 200 250 300 350 400 450

Time (fs)
Photon generation efficiency

$10\, PW \quad \tau_{\text{las}} = 150\, fs \quad \lambda = 1\, \mu m \quad \varnothing = 2.5\, \mu m \quad L_{pl} = 80\, \mu m \quad \text{(ELI-BL L4)}$

![Graph showing energy vs. time for different materials: H, Au, Fe, Fe tilt.](image)
Multi-parametric analysis of target & laser pulse
3D simulations of gamma flare generation

10 PW, 50 fs, 40 micron corona
**Kinematics of inverse multi-photon Compton scattering in vacuum**

**Nonlinear Thomson scattering** \( \hbar \omega_\gamma \approx 0.3 \hbar \omega_0 a_0^3 \) i.e. \( N_{ph} = a_0^3 \)

Recoil at \( \hbar \omega_\gamma \approx mc^2 a_0 \) imposes constraint on the photon energy:

\[
a_0 < a_m = \sqrt{m_e c^2 / \hbar \omega_0}
\]

The energy - momentum conservation equates the sum of electron and photon energy - momentum before and after scattering

\[
m_e c^2 \gamma + \hbar \omega_\gamma = m_e c^2 \gamma_0 + N_{ph} \hbar \omega_0, \quad p + \hbar \mathbf{k}_\gamma = p_0 + N_{ph} \hbar \mathbf{k}_0
\]

In vacuum \( \omega^2 = k^2 c^2 \). It yields

\[
\hbar \omega_\gamma = \frac{N_{ph} \hbar \omega_0 (p_{||0} c + m_e c^2 \gamma_0)}{N_{ph} \hbar \omega_0 + m_e c^2 \gamma_0 - p_{\perp0} c \sin \theta + (\hbar \omega_0 - p_{||0} c) \cos \theta}
\]

Maximum photon energy \( \hbar \omega_\gamma \approx m_e c^2 p_{||0} \) energy in the head - on collision.

In the receding configuration, for co - propagating electron and laser pulse the scattered photon energy is well bellow the incident photon energy.

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Inverse multi-photon Compton scattering in collisionless plasma

The energy-momentum conservation equates the sum of electron and photon energy-momentum before and after scattering:

\[ m_e c^2 \gamma + \hbar \omega_{\gamma} = m_e c^2 \gamma_0 + N_{ph} \hbar \omega_0, \quad \mathbf{p} + \hbar \mathbf{k}_{\gamma} = \mathbf{p}_0 + N_{ph} \hbar \mathbf{k}_0 \]

In collisionless plasma \( \omega^2 = k^2 c^2 + \bar{\omega}_{pe}^2 \) where \( \bar{\omega}_{pe}^2 = 4 \pi ne^2 / m_e \sqrt{1 + a_0^2} \)

Assuming \( \omega_0 = \bar{\omega}_{pe} \) (electron-e.m. wave interaction at critical surface) we obtain:

\[ \hbar \omega_{\gamma} = \frac{N_{ph} \hbar \omega_0 (\hbar \omega_0 + m_e c^2 \gamma_0)}{N_{ph} \hbar \omega_0 + m_e c^2 \gamma_0 - p_{\perp,0} c \sin \theta - p_{\parallel,0} c \cos \theta} \]

Maximum photon energy \( \hbar \omega_{\gamma} \approx m_e c^2 \gamma_0 \).

Electron momentum: \( p_{\perp,0} = m_e c a_0, \quad p_{\parallel,0} = m_e c a_0^2 / 2 \)

Gamma photons are emitted at the angle \( \theta_a \):

\[ -a_0 / p_{\parallel,0} < \theta_a < a_0 / p_{\parallel,0} \]

being confined within the angle:

\[ \Delta \theta_a \approx \sqrt{(1 + a_0^2) m_e^2 c^2 / p_{\parallel,0}^2 + N_{ph} \hbar \omega_0 / m_e c} \]
Nonlinear Electromagnetic Waves in the QED Vacuum
QFT: Quantum Electrodynamics

- Vacuum is not a void medium
- Creation and annihilation of electron-positron pairs
- Casimir effect, Delbruck scattering, photon-photon-scattering, …

- Fluctuations

In the QED, photon-photon scattering occurs via creation-annihilation of virtual electron-positron pairs by two initial photons followed by annihilation of the pairs into final photons.

\[ \sigma_{\gamma\gamma} = \begin{cases} \left(\frac{973}{10125\pi}\right) \alpha^2 r_e^2 \left(\frac{\hbar\omega_\gamma}{m_e c^2}\right)^6 & \text{for } \hbar\omega_\gamma << m_e c^2 \\ \left(\frac{3}{12\pi}\right)^2 \alpha^2 r_e^2 \left(\frac{m_e c^2}{\hbar\omega_\gamma}\right)^2 & \text{for } \hbar\omega_\gamma >> m_e c^2 \end{cases} \]

Electron-positron pair creation via the Breit-Wheeler process has the cross section:

\[ \sigma_{\gamma\gamma \rightarrow ep} = \frac{1}{2} \pi r_e^2 (1 - \beta_e^2) \left\{ (3 - \beta_e^4) \ln \left( \frac{1 + \beta_e}{1 - \beta_e} \right) - 2 \beta_e (2 - \beta_e^2) \right\} \]

where

\[ \beta_e = \sqrt{1 - \frac{2m^2 c^4}{\hbar^2 \omega_1 \omega_2 (1 - \cos \varphi)}} \]

is Lorentz invariant.

Near the threshold when \( \beta_e \leq 1 \)

\[ \omega_1 \omega_2 (1 - \cos \varphi) \geq 2m^2 c^4 / \hbar^2 \]

the cross section is given by

\[ \sigma_{\gamma\gamma \rightarrow ep} = \pi r_e^2 \sqrt{\frac{1}{\beta_e} - 1} \]

G. Breit and J. A. Wheeler, Physical Review 46, 1087 (1934)
X Ribeyre, E d’Humi`eres, S Jequier and V T Tikhonchuk, PPCF 60, 104001 (2018)
Photon-Photon Scattering: the Number of Scattered Photons

The number of photons in the electromagnetic pulse with amplitude $E$ in the 4-volume $\lambda^3/\omega$

$$N_\gamma = \frac{E^2 \lambda^3}{4\pi \hbar \omega}$$

In optimal regime (maximal luminosity) the number of scattering events per 4-volume is proportional to the scattering cross section, to the product of the photon numbers in colliding photon bunches, and it is inverse proportional to the square of the wavelength

$$N_{\gamma\gamma} = \sigma_{\gamma\gamma} \frac{N_\gamma N_\gamma}{\pi \lambda^2}$$

This yields

$$N_{\gamma\gamma} = \left(\frac{973}{10125\pi}\right) \alpha^2 \left(\frac{E}{E_s}\right)^4 \propto \left(\frac{\alpha}{I_s}\right)^2$$

Here $E_s$ is the Schwinger (Sauter, Bohr, …) field equal to

$$E_s = \frac{m_e^2 c^3}{e \hbar}$$

where $\alpha = e^2 / \hbar c \approx 1/137$ : e.m. field intensity $I_s = cE_s^2 / 4\pi = 10^{29} W / cm^2$

QED VACUUM IS DISPERSIONLESS MEDIUM
Heisenberg-Euler Lagrangian

In the long wavelength and low frequency approximation ($|\partial_{\mu}A_{\nu}|/|A_{\mu}|<\lambda^{-1}$), the Lagrangian describing the electromagnetic field in vacuum is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}',$$

where

$$\mathcal{L}_0 = -\frac{1}{16\pi} F_{\mu\nu}^\ast F_{\mu\nu}$$

with

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

gives the Maxwell equations.

The Heisenberg-Euler term

$$\mathcal{L}' = -\frac{m^4}{8\pi^2} \int_0^{\infty} \frac{\exp(-\eta)}{\eta^3} \left\{ 1 - \frac{\eta^2}{3} (a^2 - b^2) - \left[ \eta a \cot(\eta a) \right] \left[ \eta b \coth(\eta b) \right] \right\} d\eta$$

Here the invariants $a$ and $b$ can be expressed in terms the Poincare invariants

$$\mathcal{S} = F_{\mu\nu}^\ast F_{\mu\nu} \quad \text{and} \quad \mathcal{G} = F_{\mu\nu} F_{\mu\nu}$$

(dual tensor equals $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$) as

$$a = \sqrt{\mathcal{S}^2 + \mathcal{G}^2 + \mathcal{S}} \quad \text{and} \quad b = \sqrt{\mathcal{S}^2 + \mathcal{G}^2 - \mathcal{S}}$$

Here we use the units $c = \hbar = 1$, and the e.m. field is normalized on the QED critical field $E_S$.

W. Heisenberg and H. Euler, Zeit. fuer Phys. 98, 714 (1936)
The Heisenberg-Euler term in the weak field approximation we have

\[ \mathcal{L}' = \frac{\kappa}{4} \left\{ \left( F^{\mu\nu} F_{\mu\nu} \right)^2 + \frac{7}{4} \left( F^{\mu\nu} \tilde{F}_{\mu\nu} \right)^2 + \frac{90}{315} \left( F^{\mu\nu} F_{\mu\nu} \right) \left[ \left( F^{\mu\nu} F_{\mu\nu} \right)^2 + \frac{13}{16} \left( F^{\mu\nu} \tilde{F}_{\mu\nu} \right)^2 \right] \right\} \]

With the constant

\[ \frac{\kappa}{4} = \left( \frac{e^4}{360\pi^2} \right) m_e^4 \]

The first two terms describe four interacting photons and the last two terms correspond to six photon interaction.

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}' \] gives equations of the nonlinear electrodynamics in the dispersionless medium with refraction index depends on the electromagnetic field.
Dirac’s Light Cone Coordinates

Below we use the light cone coordinates

\[ x^+ = (x + t) / \sqrt{2}, \quad x^- = (x - t) / \sqrt{2} \]

Here \( c = 1 \).

The variables \((x, t)\) in the lab. frame of reference are related to \((x', t')\) in the boosted frame moving with the velocity \( \beta \) as

\[ x' = x \cosh \eta - t \sinh \eta, \quad t' = t \cosh \eta - x \sinh \eta \]

where \( \eta = \sqrt{(1 + \beta)/(1 - \beta)} \)

The Lorentz transform of the light-cone variables is

\[ x'^+ = (x' + t') / \sqrt{2} = e^{-\eta} (x + t) / \sqrt{2} = e^{-\eta} x^+, \quad x'^- = (x' - t') / \sqrt{2} = e^{+\eta} (x - t) / \sqrt{2} = e^{+\eta} x^- \]

For derivatives we have \((\partial_\pm)' = e^{\pm\eta} \partial_\pm\)

Introducing the fields \( u = \partial_- a \) and \( w = \partial_+ a \)

related to the electric \( e = -\partial_t a \) and magnetic \( b = \partial_x a \) field by

\[ u = (e - b) / \sqrt{2} \quad \text{and} \quad w = (e + b) / \sqrt{2} \]

we obtain

\[ u' = e^{-\eta} u \quad \text{and} \quad w' = e^{+\eta} w \]

The field product \( uw = (e^2 - b^2) / 2 \)

\[ u'w' = uw \]

is Lorentz invariant.
Finite Amplitude Counter-Propagating EM Waves

The e.m. waves have the same polarization \( A = Ae_z \)

We use the light cone coordinates

\[
x^+ = (x + t)/\sqrt{2}, \quad x^- = (x - t)/\sqrt{2}
\]

Propagation along the \( x \) axis,

\[
A = Wx^+ + a(x^+, x^-)
\]

where \( Wx^+ \) is the crossed e.m. field

and \( a(x^+, x^-) \) is the 4 potential for counter-propagating e.m. wave.

H-E Lagrangian can be cast to

\[
\mathcal{L} = -\frac{1}{4} \left[ (W + w)u - \epsilon_2 (W + w)^2 u^2 - \epsilon_3 (W + w)^3 u^3 \right]
\]

with the field components \( u = \partial_+ a, \quad w = \partial_- a \)

Lagrangian variation

\[
\partial_+ (\partial L/\partial u) + \partial_- (\partial L/\partial w) = 0
\]

yields system of nonlinear equations for electromagnetic wave

\[
\partial_+ w - \partial_- u = 0
\]

\[
\left[ 1 - 4\epsilon_2 (W + w)u - 9\epsilon_3 (W + w)^2 u^2 \right] \partial_+ u - \left[ \epsilon_2 (W + w)^2 + 3\epsilon_3 (W + w)^3 u \right] \partial_- u
\]

\[
-\left[ \epsilon_2 u^2 + 3\epsilon_3 (W + w)u^3 \right] \partial_+ w = 0
\]

with \( \epsilon_2 = (2/45\pi)\alpha \) and \( \epsilon_3 = (32/315\pi)\alpha \)
Phase and Group Velocity

The e.m. wave propagates along the $x$ direction with velocity

$$v_W = \frac{1 - \varepsilon_W^2}{1 + \varepsilon_W^2} \approx 1 - 2\varepsilon_W^2$$

The phase velocity of the wave is equal to the group velocity, i.e. the vacuum is dispersionless medium.

The normalized $\beta_{ph} = \beta_g = v_W / c$ are less than unity, i.e., the wave phase (group) velocity is below the speed of light in vacuum.

Corresponding dispersion equation for the e.m. wave frequency and wave number has a form

$$(\omega + kc)(\omega - kc + 2kc\varepsilon_W^2) = 0$$

Refraction index of the QED vacuum $n = 1 + 2\varepsilon_W^2$ is greater than unity with $\Delta n = 2\varepsilon_W^2$

Bialynicka - Birula & Bialynicki - Birula (1970)
Dittrich & Gies (2000)
Marklund & Shukla (2006)
Electromagnetic Riemann Wave in Vacuum

Assuming the solution of nonlinear wave equations in the Riemann wave form: \( w = w(u) \), we obtain

\[
\left[ \varepsilon_2 u^2 + 3\varepsilon_3 (W + w) u^3 \right] J^2 - \left[ 1 - 4\varepsilon_2 (W + w) u - 9\varepsilon_3 (W + w)^2 u^2 \right] J + \left[ \varepsilon_2 (W + w)^2 + 3\varepsilon_3 (W + w)^3 u \right] = 0
\]

Here \( J = \frac{dw}{du} \) is the Jacobian.

For small but finite \( u \) it can be found to be equal to

\[
J = \varepsilon_2 W^2 + \left( 4\varepsilon_2^2 + 3\varepsilon_3 \right) W^3 u + \ldots
\]

We arrive to the equation for nonlinear e.m. wave in the form

\[
\partial_+ u + \left( \varepsilon_2 W^2 + \left( 4\varepsilon_2^2 + 3\varepsilon_3 \right) W^3 u \right) \partial_- u = 0
\]

In the \( x, t \) variables it can be rewritten as

\[
\partial_+ u + \left( v_w - 2 \left( 4\varepsilon_2^2 + 3\varepsilon_3 \right) W^3 u \right) \partial_+ u = 0
\]
Wave Breaking (Rarefaction Wavebreak)

This is the Hopf equation. We rewrite it as

\[ \partial_t \bar{u} + (v_w - \bar{u}) \partial_x \bar{u} = 0 \]  

(*)

with

\[ \bar{u} = 2 \left( 4\varepsilon_2^2 + 3\varepsilon_3 \right) W^3 u \]

For given initial condition \( \bar{u}_0(x) \) its solution in implicit form is given by

\[ \bar{u} = \bar{u}_0 \left( x - (v_w - \bar{u})t \right) \]

(**)

It can be obtained by integrating along the characteristics:

\[ \frac{dx}{dt} = v_w - \bar{u}, \quad \frac{d\bar{u}}{dt} = 0 \]

with the solution \( \bar{u} = \bar{u}_0(x_0) \), \( x = x_0 + (v_w - \bar{u}_0(x_0))t \)

Calculating the field gradient we find that

\[ \partial_x u = \frac{\partial_x u_0(x_0)}{1 - 2 \left( 4\varepsilon_2^2 + 3\varepsilon_3 \right) W^3 \partial_x u_0(x_0)t} \]

At \( t = t_{br} \)

\[ t_{br} = \frac{1}{2 \left( 4\varepsilon_2^2 + 3\varepsilon_3 \right) W^3 \partial_x u_0(x_0)} \]

the denominator vanishes - the wave breaks (gradient catastrophe)

Lutzky, Toll, Phys. Rev. 113, 1649 (1959); Böhl, King, Ruhl, Phys. Rev. A 92, 032115 (2015);
Expression (**) describes high order harmonics generation and wave steepening in a vacuum. Perhaps one of the simplest mechanisms. We choose the initial electromagnetic wave as

\[
\vec{u}_0 = \vec{a}_1 \cos (kx)
\]

and find from (**) that \( \vec{u}(x, t) = \vec{a}_1 \cos (k(x - v_w t)) - \frac{1}{2} \vec{a}_1^2 k v_w t \sin (2k(x - v_w t)) + \ldots \)

When a wave approaches the wave breaking point, wave steepening is equivalent to harmonics generation with higher numbers leading to the gradient catastrophe, i.e., at the shock wave.

The e.m. shock wave separates two regions, I and II, where the function \( u(x, t) \) takes the values \( u_I \) and \( u_{II} \). The shock wave front (it is an interface between regions I and II) moves with a velocity equal to \( v_{sw} \), i.e. shock wave front is localized at \( x_{sw} = v_{sw} t \).

Integrating Eq. (*) over an infinitely small interval \( (x_{sw} - \delta, x_{sw} - \delta) \), where \( \delta \to 0 \), we obtain

\[
\left\{-v_{sw} u + v_w u - (4\varepsilon_2^2 + 3\varepsilon_3) W^5 u^2 \right\}_{x=x_{sw}} = 0
\]

where \( \{f\}_x = f(x + \delta) - f(x - \delta) \) at \( \delta \to 0 \) denotes the discontinuity of function \( f(x) \) at point \( x \).

This yields for the shock wave front velocity

\[
v_{sw} = v_w - (4\varepsilon_2^2 + 3\varepsilon_3) W^5 (u_I + u_{II})
\]

Shock wave front width should be of the order of the Compton wavelength, \( \lambda_C = \hbar / m_e c \).
L’Oréal – UNESCO award for women in science 2019, awarded to Hedvika Kadlecová

H. Kadlecová, et al., *Electromagnetic shocks in the quantum vacuum*  

“Hedvika Kadlecová who works in the ELI Beamlines Research Center in Dolní Břežany became one of the three winners of the prestigious award L’Oréal-UNESCO for women in science for her theoretical work on nonlinear electromagnetic waves in quantum vacuum”

Nonlinear equations for electromagnetic wave in 1D case can be written as

$$\partial_+ w - \partial_- u = 0$$

$$\left[ 1 - 4\varepsilon_2 uw - 9\varepsilon_3 u^2 w^2 \right] \partial_+ u - \left[ \varepsilon_2 w^2 + 3\varepsilon_3 w^3 u \right] \partial_- u = \left[ \varepsilon_2 u^2 + 3\varepsilon_3 u^3 w \right] \partial_+ w$$

with $\varepsilon_2 = (2/45\pi)\alpha$ and $\varepsilon_3 = (32/315\pi)\alpha$

System is linear with respect to highest order terms in partial derivatives $\partial_+$ and $\partial_-$ with coefficients nonlinearly dependent on $u$ and $w$, admits hodograph transformation. We consider $x_-$ and $x_+$ as functions of $u$ and $w$. This yields

$$\partial_u x_+ - \partial_w x_- = 0$$

$$\left[ 1 - 4\varepsilon_2 uw - 9\varepsilon_3 u^2 w^2 \right] \partial_w x_- + \left[ \varepsilon_2 w^2 + 3\varepsilon_3 w^3 u \right] \partial_w x_+ + \left[ \varepsilon_2 u^2 + 3\varepsilon_3 u^3 w \right] \partial_u x_+ = 0$$

It possesses reach family of particular solutions including Lorentz invariant solutions:

$$x^+ = \frac{C}{u(1-2\varepsilon_2 uw-3\varepsilon_3 u^2 w^2)} \quad \text{and} \quad x^- = \frac{C}{w(1-2\varepsilon_2 uw-3\varepsilon_3 u^2 w^2)}$$
Discussions

**Experiment:** Measuring the phase difference between the phase of the electromagnetic pulse colliding with the counterpropagating wave and the phase of the pulse which does not interact with high intensity wave, it is equal to

\[
\delta \psi = 4\pi \frac{d}{\lambda} \varepsilon W^2
\]

where \(\lambda\) is the wavelength of high frequency pulse and \(d\) is the interaction length, plays a central role in discussion of experimental verification of the QED vacuum birefringence.


For 10 PW laser the radiation intensity can reach \(10^{24}\) W/cm\(^2\), for which \(W^2 = 10^{-5}\).

Taking the ratio \(d / \lambda\) equal to \(10^4\), i.e. equal to the ratio between the optical and x-ray radiation wavelength, we find that \(\delta \psi = 10^{-4}\).

**HOHG:** Assuming \(k\nu_w t = 2\pi d / \lambda\) and the intensity of the x-ray pulse of \(10^{21}\) W/cm\(^2\), we obtain that the ratio of the second harmonic amplitude to the amplitude of fundamental harmonic is approximately equal to \(10^{-11}\).
Synergic
Cherenkov Radiation –
Compton Scattering

\[ \theta_c = \text{ArcCos}^{-1} \left( \frac{c}{vn} \right) \]
At the focus of 10 PW laser the field intensity can reach $10^{24}$ W/cm$^2$, i.e. $a_0 = 10^3$.

Vacuum polarization changes the refraction index: radiation correction to the "photon mass" results in the dispersion equation for the e.m. wave frequency and wave vector

$$\omega^2 - k^2 c^2 - \mu^2_{||,\perp} \frac{c^2}{\hbar^2} = 0$$

where (Ritus 1970)

$$\mu^2_{||,\perp} = -\alpha m_e^2 \left\{ \begin{array}{ll}
\frac{11 \pm 3}{90 \pi} \chi^2_{\gamma} + i \sqrt{\frac{3}{2}} \frac{3 \pm 1}{16} \chi_{\gamma} \exp\left(-\frac{8}{3 \chi_{\gamma}}\right) & \text{for } \chi_{\gamma} \ll 1 \\
\frac{5 \pm 1}{28 \pi^2} \sqrt{3 \Gamma^4} \left(\frac{2}{3}\right) (1-i\sqrt{3}) \left(3 \chi_{\gamma} \right)^{2/3} & \text{for } \chi_{\gamma} \gg 1
\end{array} \right.$$ 

When $\alpha \chi_{\gamma}^{2/3} \to 1$ the "photon mass" tends to $m_e$.

In the limit $\chi_{\gamma} \ll 1$ the difference between the vacuum refraction index and unity is

$$\Delta n_{\pm} = \alpha \frac{11 \pm 3}{45 \pi} \left(\frac{E}{E_S}\right)^2$$

with $E_S = m_e^2 c^3 / e h$ i.e. the normalized phase velocity of the e.m. wave equals

$$\beta_{\pm} = 1 - \varepsilon_{\pm} \left(\frac{E}{E_S}\right)^2$$

where $\varepsilon_{\pm} = \alpha (11 \pm 3) / 45 \pi \approx 10^{-4}$.
Following to **V. L. Ginzburg (1940)** we describe the kinematics of electron-photon interaction in QED vacuum as

\[ p_0 + \hbar k_0 = p + \hbar k, \quad m_e c^2 \gamma_0 + \hbar \omega_{\gamma,0} = m_e c^2 \gamma + \hbar \omega_{\gamma} \]

with \( \gamma_0 = \sqrt{1 + p_0^2 / m_e^2 c^2}, \quad \gamma = \sqrt{1 + p^2 / m_e^2 c^2}, \quad p = p_\parallel e_x + p_\perp e_y, \) and \( k = |k| (\cos \theta e_x + \sin \theta e_y) \)

In medium with \( n \neq 1 \) the wave frequency \( \omega \) and wave vector \( k \) are related to each other as \( k = \frac{k}{|k| c} n_\pm \omega. \)

The photon energy is

\[ \hbar \omega_{\gamma} = g \pm \sqrt{g^2 + 2sh \omega_0 \left( \frac{m_e c^2 \gamma_0 + p_\parallel c}{n_\pm^2 - 1} \right)} \]

where \( s \) is the number of photons and

\[ g = \left( p_\parallel c - sh \omega_0 \right) n_\pm \cos \theta - m_e c^2 \gamma_0 - sh \omega_0 \]

When the function \( g \) is positive and \( sh \omega_0 \ll m_e c^2 / \gamma_0 \) the photon energy can be found to be

\[ \hbar \omega_{\text{Ch}} \approx 2g + sh \omega_0 \left( \frac{m_e c^2 \gamma_0 + p_\parallel c}{2(n_\pm^2 - 1)} \right) \]

This corresponds to the Cherenkov radiation.

In the opposite limit, when \( g < 0 \), in the limit \( sh \omega_0 \ll m_e c^2 / \gamma_0 \) the Compton scattering mode frequency is given by

\[ \hbar \omega_c \approx \frac{sh \omega_0 \left( m_e c^2 \gamma_0 + p_\parallel c \right)}{\left( p_\parallel c - sh \omega_0 \right) n_\pm \cos \theta - m_e c^2 \gamma_0 - sh \omega_0} \]

In the limit \( s < s_m \), where \( s_m = m_e c^2 / 4 \hbar \omega_0 \gamma_0 \) the photon energy equals \( \hbar \omega_c = 4sh \omega_0 \gamma_0^2 \); at \( s > s_m \), we have \( \hbar \omega_c = m_e c^2 \gamma_0^2 \).
Cherenkov radiation

Condition of Cherenkov radiation,

\[ sh\omega_0 \ll \frac{\sqrt{m_e^2 c^4 + p_{\parallel,0}^2 c^2} - p_{\parallel,0} c n_\pm \cos \theta}{1 + n_\pm \cos \theta} \]

The electron energy should be large enough to have

\[ \gamma_0 > \gamma_{Ch} = \frac{1}{\sqrt{2\Delta n_\pm}} = \frac{\sqrt{45\pi E_S^2}}{\alpha (11 \pm 3) E_0^2} \approx 30 \frac{I_S}{I_0} \]

Here the laser intensity \( I_0 = cE_0^2 / 4\pi \) in the focus region of 10 PW laser is approximately equal to \( 10^{24} \) W/cm\(^2\); \( I_S = cE_S^2 / 4\pi \approx 10^{29} \) W/cm\(^2\), i. e. the Cherenkov radiation threshold is exceeded for the electron energy above 10 GeV.

The Cherenkov cone with the angle \( \theta_{Ch} = 2\sqrt{\varepsilon_\pm I_0 / I_S} \) in the focus of 10 PW laser it is approximately equal to \( 2 \times 10^{-5} \).

The rate of the energy loss due to the Cherenkov radiation friction force along the electron trajectory is

\[ \frac{d\mathcal{E}_e}{dx} = -\frac{e^2}{c^2} \int_{v_e n_\pm / c > 1} \left( 1 - \frac{c}{v_e n_\pm} \right) \omega d\omega \approx -\frac{e^2}{\kappa_C^2} \varepsilon_\pm \left( \frac{E_0}{E_S} \right)^2 \]

Integration is done over the region where \( v_e n_\pm / c > 1 \).

The formation length is given by

\[ l_{Ch} \approx \kappa_C \gamma_e \quad (\kappa_C = \hbar / m_e c = 3.8 \times 10^{-11} \text{cm}) \]

It is approximately equal to \( 2 \times 10^{-5} \).

Traversing the laser focus region the electron emits 0.2 photons.

For the electric charge of the LWFA electron bunch of 100 pC we obtain \( 10^4 \) photons.

As one may see from the expression for the photon invariant mass

\[ \mu_{\parallel,\perp}^2 = -\alpha m_e^2 \left\{ \begin{array}{ll}
11 \pm 3 \sqrt{\frac{3}{2}} \frac{3 \pm 1}{16} \chi_{\gamma} \exp \left( -\frac{8}{3 \chi_{\gamma}} \right) & \text{for } \chi_{\gamma} \ll 1 \\
\frac{5 \pm 1}{28 \pi^2} \sqrt{3} \Gamma^4 \left( \frac{2}{3} \right) \left( 1 - i\sqrt{3} \right) (3 \chi_{\gamma})^{2/3} & \text{for } \chi_{\gamma} \gg 1
\end{array} \right. \]

at the high photon energy end, when the parameter \( \chi_{\gamma} \) becomes larger than unity, the vacuum polarization effects weaken. In this limit the Cherenkov radiation does not occur. As a result the photons with the energy above \( \hbar \omega_{\gamma} = m_e c^2 E_S / E_0 \) are not present in the high frequency spectrum of the radiation. For 10 PW laser parameters this energy is approximately equal to 100 MeV.
Towards Schwinger Limit

\[ I_s = 10^{29} \frac{W}{cm^2} \quad \Rightarrow \quad P = 10^{21} \left( \frac{\lambda}{1\mu m} \right)^2 W \quad \Rightarrow \quad E = 3 \times 10^6 \left( \frac{\lambda}{1\mu m} \right)^3 J \]
Relativistic Flying Mirror Concept

\[ a = \frac{eE}{m_e e \omega} = \text{inv} \]
\[ \gamma_{ph} = \frac{1}{\sqrt{1 - \beta_{ph}^2}} \]
\[ v_e = v_{ph} \]

\[ \lambda' = \lambda_s \frac{\sqrt{1 - \beta_{ph}}}{\sqrt{1 + \beta_{ph}}} \approx \frac{\lambda_s}{2\gamma_{ph}} \]
Focal spot diameter \( \approx \lambda' \)

\[ L_s' \approx \frac{L_s}{2\gamma_{ph}} \]
EM Pulse Length:

\[ \lambda'' = \lambda_s \frac{1 - \beta_{ph}}{1 + \beta_{ph}} \approx \frac{\lambda_s}{4\gamma_{ph}^2} \]
Focal region: \( l_{\perp} \approx \lambda' \)
\( l_{\parallel} \approx \lambda'' \)
Relativistic Flying Mirror: Concept & Proof-of-Principle

Concept:

\[ \omega_r = \omega_0 \frac{1 + \beta_M}{1 - \beta_M}, \quad I_r = I_0 \gamma^6 \frac{R (D / \lambda_0)}{2} \]

Experiments

Results

Space-Time Overlapping of Driver and Source Pulses

Vacuum focus 630 μm Jet center

t_p = 0

Driver 200 mJ, 76 fs Source 12 mJ

In a plane EM wave, both the invariants \[ \mathcal{S} = F^{\mu\nu} F_{\mu\nu} \quad \& \quad \mathcal{G} = F^{\mu\nu} \tilde{F}_{\mu\nu} \] are zero, therefore \( e^-e^+ \) pairs are not created for an arbitrary magnitude of the EM field.

Relativistic flying mirror can create field \( \gamma_M \) times greater than QED critical field.
The driver and source must carry 10 kJ and 30 J, respectively.

Reflected intensity can approach the Schwinger limit

\[ I_{QED} = 10^{29} \text{W/cm}^2 \]

It becomes possible to investigate such the fundamental problems of nowadays physics, as e.g. the electron-positron pair creation in vacuum and the photon-photon scattering

\[ P_{cr} = \frac{45\pi^2}{\alpha} \frac{cE_s^2\lambda^2}{4\pi} \]

The critical power for nonlinear vacuum effects is

\[ P_{cr} \approx 2.5 \times 10^{24} \text{W} \]

Light compression and focusing with the FLYING MIRRORS yields

\[ P = P_0 \gamma_{ph} \]

for \( \lambda_0 = 1\mu m \), \( \lambda = \lambda_0 / 4\gamma_{ph}^2 \) with \( \gamma_{ph} \approx 30 \) the driver power \( P_{cr} = 100PW \)

Laser Driven $e^-e^+\gamma$ Plasma
• Sub-MeV $e^+$ from radioactive isotopes or accelerators are used in material science:
  A. P. Mills, Science 218, 335 (1982); P. J. Schultz and K. G. Lynn, Rev. Mod. Phys. 60, 701 (1988);

• $e^+$ emission tomography:

• Basic antimatter science such as antihydrogen experiments:
  M. Amoretti, et al., Nature (London) 419, 456 (2002);

• Bose-Einstein condensed positronium:

• Basic plasma physics:
In the **Leptonic Era** the Universe consisted mainly of leptons and photons. It began when the temperature dropped below $10^{12}$ K some $10^{-4}$ seconds after the Big Bang, and it lasted until the temperature fell below $10^{10}$ K, at an era of about 1 second.

e\textsuperscript{-} - e\textsuperscript{+} pairs via trident + Bethe-Heitler processes

- \textbf{e\textsuperscript{-} - e\textsuperscript{+} generation in laser-solid target interaction (trident mechanism of the e\textsuperscript{-} - e\textsuperscript{+} pair generation)}

\[ \sigma_{e^-Z \rightarrow e^+e^-Z} = \frac{(14/27\pi)r_e^2(\alpha Z)^2 \ln \left(1 + p^2/m_e^2c^2\right)}{27\pi} \]

\[ r_e = \frac{e^2}{m_e c^2}, \quad \alpha = \frac{e^2}{\hbar c} = 1/137 \]

Breit-Wheeler process of \( e^- - e^+ \) generation

\[ \hbar \omega + \hbar \omega' \rightarrow e^+ + e^- \]

with \( \sigma_{\omega \omega \rightarrow e^+ e^-} \approx \alpha^2 r_e^2 \)

if \( \hbar^2 \omega \omega' > m_e^2 c^4 \)

Multiphoton inverse Compton &

Multiphoton B-W processes

\[ e^- + N\hbar \omega_0 + \rightarrow \hbar \omega_\gamma + e^- \]

\[ \hbar \omega_\gamma + N\hbar \omega_0 \rightarrow e^+ + e^- \]

50 GeV

1.6 ps \( a_0 \approx 0.3 \)

\[ \chi_e = \left( \frac{E}{E_S} \right) \gamma_e \approx 0.3 \]

\[ \chi_\gamma = \left( \frac{E}{E_S} \right) \left( \frac{\hbar \omega_\gamma}{m_e c^2} \right) \approx 0.15 \]

G. Breit, J. A. Wheeler, Phys. Rev. 46, 1087 (1934)
Electromagnetic cascades induced by pairs

\[ \chi_e = \frac{e^2 \hbar \sqrt{(F_{\mu\nu} p_{\mu})^2}}{m_e c^4} = \gamma_e \frac{E}{E_S} \gg 1 \quad \rightarrow \quad n_+ \approx n_- = \frac{E}{4\pi e\lambda_0} \]

\[ I > 2.5 \times 10^{24} W/cm^2 \]

D.L. Burke et al., PRL 79, 1626 (1997)

\[ \chi_e \approx 0.3, \quad \chi_\gamma \approx 0.15 \]

- W. Heisenberg, H. Euler (1936)
Probability of pair creation

$$W(\chi_\gamma) = \frac{3}{32} \frac{e^2 m_e^2 c^3}{\hbar^3 \omega_\gamma} \left( \frac{\chi_\gamma}{2\pi} \right)^{3/2} \exp \left( -\frac{8}{3 \chi_\gamma} \right) \quad \text{for } \chi_\gamma \ll 1$$

$$W(\chi_\gamma) = \frac{27 \Gamma^7(2/3) e^2 m_e^2 c^3}{56\pi^5 \hbar^3 \omega_\gamma} \left( \frac{3 \chi_\gamma}{2} \right)^{2/3} \quad \text{for } \chi_\gamma \gg 1$$

The number of absorbed laser photons :

$$N_i \gg a$$

Photon mean - free - path before the pair is created :

$$l_{m.f.p} = \frac{\lambda_0}{0.2\pi \alpha a} \approx 220 \frac{\lambda_0}{a}$$

**THRESHOLD** : $\chi_\gamma > 1$ for $\chi_\gamma \approx \chi_e$ and $\gamma_e = a$

$$\chi_e = \frac{a}{a_S} \gamma_e \approx \frac{a^2}{a_S} > 1 \quad \text{gives } a > \sqrt{a_S} \approx 10^3 \quad \text{gives } I > 10^{24} W / cm^2$$

Here $a_S = eE_S / m_e \omega c = m_e c^2 / \hbar \omega$

“Schwinger” mechanism of $e^- - e^+$ pair creation

Energy is plotted against momentum and coordinate (electric field is parallel z-axis).

Solution to Dirac equation shows that the $\psi$ function is large only in the regions I and III.

In the region II, it decreases exponentially.

Therefore transmission coefficient through region II, which gives probability of pair creation, has the order of magnitude $\exp(-\pi m_e^2 c^3 / |e| \hbar E)$

Quasiclassical regime (F. Sauter, 1931)

$$w = A \exp \left( -2 \frac{z_2 = +m_e c^2 / eE}{\hbar} \int_{z_1 = -m_e c^2 / eE}^1 |p(z)| \, dz \right)$$

$$= A \exp \left( -\frac{4m_e^2 c^3}{e |E| \hbar} \int_0^1 \sqrt{1 - \xi^2} \, d\xi \right)$$

$$\propto \exp \left( -\frac{\pi E_S}{E} \right)$$

Probability (W. Heisenberg, H. Euler, 1936)

$$w = \frac{1}{4\pi^3} \left( \frac{|e| |E| \hbar}{m_e^2 c^3} \right)^2 \frac{m_e c^2}{\hbar} \left( \frac{m_e c}{\hbar} \right)^3 \exp \left( -\frac{\pi m_e^2 c^3}{|e| \hbar E} \right)$$
\[ \tilde{\mathcal{F}} = \frac{E^2 - B^2}{2} = \frac{F_{\mu\nu}^2}{4} = \text{inv}, \quad \mathcal{G} = (\mathbf{E} \cdot \mathbf{B}) = \frac{F_{\mu\nu} F_{\mu\nu}^*}{4} = \text{inv} \]

\[ \mathcal{E} = \sqrt{\mathcal{F}^2 + \mathcal{G}^2 + \mathcal{F}}, \quad \mathcal{B} = \sqrt{\mathcal{F}^2 + \mathcal{G}^2 - \mathcal{F}} \]

are the electric and magnetic fields in the frame where they are parallel

\[ W_{e^+e^-} = \frac{e^2 E_S^2}{4\pi^2 \hbar^3 c} \text{abhcoth} \left( \frac{\pi b}{a} \right) \exp \left( -\frac{\pi}{a} \right) \]

\[ b = \mathcal{B}/E_S, \quad a = \mathcal{E}/E_S \]

for \( \epsilon \to 0 \) the rate \( W_{e^+e^-} \to 0 \)

at \( \beta \to 0 \) the rate

\[ W_{e^+e^-} = \frac{e^2 E_{schw}^2}{4\pi^2 \hbar^3 c} a^2 \exp \left( -\frac{\pi}{a} \right) \]

The number of e\(^+\)e\(^-\) pairs

\[ N_{e^+e^-} = \iiint W_{e^+e^-} dVdt = \frac{c\tau l_x l_y l_z}{64\pi^4 \lambda_C^4} a_m^4 \exp \left( -\frac{\pi}{a_m} \right) \]
Electron-positron pairs can be created before the laser field reaches the Schwinger limit, due to a large phase volume occupied by a high-intensity EM field.


### Multiple 10kJ beam system provides necessary conditions for $e^-e^+$ pairs creation.

<table>
<thead>
<tr>
<th>Number of pulses</th>
<th>Number of $e^-e^+$ with 10kJ pulses</th>
<th>Required energy (kJ) to create one pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$9 \times 10^{-19}$</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>$3 \times 10^{-9}$</td>
<td>20</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>16</td>
<td>$1.8 \times 10^3$</td>
<td>8</td>
</tr>
<tr>
<td>24</td>
<td>$4.2 \times 10^6$</td>
<td>5.1</td>
</tr>
</tbody>
</table>

The vector field shows $r$- and $z$-components of the poloidal electric field in the plane $(r,z)$:

The color density shows toroidal magnetic field distribution.

The first Poincare invariant

$$\tilde{\mathcal{S}}^\text{TM} = \frac{E^2 - B^2}{2a_0^2}$$

**3D EM configuration TM - mode**

**TM configuration:**

**Magnetic field**

$$B(R, \theta) = e^{\phi} \frac{B_0 \sin(\omega_0 t)}{(8\pi k_0 R)^{1/2}} J_{n+1/2}(k_0 R) L_n^i(\cos \theta)$$

**Electric field**

$$E = i k_0^{-1} \left( \nabla \times B \right)$$

Threshold for the $e^-$ - $e^+$ avalanche

**INSTABILITY**: $e^-$ ($e^+$) trajectory in $r, z$ - plane $\omega_0 t = 1, a_0 \gg 1$

\[
p_z(t) = m_e c a_0 \omega_0 t,
\]
\[
p_r(t) = m_e c \frac{a_0 k_r r_0 \omega_0 t}{2^{3/2}} I_1 \left( \frac{\omega_0 t}{2^{3/2}} \right),
\]
\[
r(t) = \frac{a_0 r_0}{2^{3/2}} I_1 \left( \frac{\omega_0 t}{2^{3/2}} \right) + \frac{a_0 r_0 \omega_0 t}{16} \left[ I_0 \left( \frac{\omega_0 t}{2^{3/2}} \right) + I_2 \left( \frac{\omega_0 t}{2^{3/2}} \right) \right]
\]

with $k_r r_0 = (2.5 a_0 / \pi a_s)^{1/2}$; growth rate: $\Gamma = \omega_0 / 2^{3/2}$

**THRESHOLD**: assuming $\omega_0 t \gg 0.1 \pi$ we find that the avalanche will start at $a_0 / a_s \gg 0.105$

which corresponds to $I^* \gg 4 \times 10^{27} \text{ W/cm}^2$
The pairs are created in a small 4-volume near the electric field maximum (maximum of $\mathcal{F}_{\text{TM}}$) with the characteristic size

$$\pi r_0^2 z_0 c t_0 = \frac{5^{3/2} \lambda_0^4}{16\pi^5} \left( \frac{a_0}{a_s} \right)^2$$

Here normalized Schwinger field is

$$a_s = \frac{m_e c^2}{\hbar \omega_0} = 5.1 \times 10^5$$

Integrating over the 4-volume the probability of the pair creation we obtain the number of pairs produced per wave period,

$$N_{e^+e^-} = \iiint W_{e^+e^-} dV dt = \frac{5^{3/2}}{4\pi^3} a_0^4 \exp \left( -\pi \frac{a_s}{a_0} \right)$$

The first pairs can be observed for an one-micron wavelength laser intensity of the order of $\approx 10^{27} \text{ W/cm}^2$, which corresponds to $a_0 / a_s = 0.05$, i.e. a characteristic size, $r_0$, is approximately equal to $0.04 \lambda_0$. 
Thank you for listening to me!